STEMkey

Teaching standard STEM topics with a key competence approach

STEMkey Module O2

Functions

Co-funded by the Erasmus+ Programme of the European Union

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Summary

This module focuses on the standard mathematical topic of functions. Functions are a fundamental concept in mathematics and they are needed for the mathematization of many types of problems starting from early years in education. For example, physical laws are modelled by functions, as well as some behaviour of biological systems. They can be used "to understand phenomena, to predict developments, or to optimize processes" (Drijvers, et al., 2019). However, in mathematics teaching, functions are mostly tackled only with a small number of real-life (interdisciplinary or transdisciplinary) examples and the introduced concepts from various subjects often remain isolated and not connected to mathematics. Thus, in this module we aim to support teaching of this standard topic within the frame of a key competence approach by extracting the concept from various contexts that occur in real-life situations and in standard topics in other school subjects. The purpose of the module is also to open a discussion with mathematics teachers about the teaching and learning potential of these rich situations.

We start off from open realistic situations that invite students to mathematically model them. We adopt the description of mathematical modelling as "applied mathematical problem solving" (Blum, 1993), which dominantly "involves connecting mathematics and the world around us, applying mathematics and inventing mathematics to solving problems". Therefore, it "is an indispensable element in sense making mathematics education" (Drijvers et al., 2019). To deal with functions in this module, the approach of inquiry-based learning will be used (Dorier & Maass, 2014, Bruder & Prescott 2013). On a meta-level we will also discuss how to work with open tasks and how to orchestrate the classroom activities that support students' construction of knowledge, investigation of strategies and presentation skills. Open tasks support the involvement of all students as active learners and are suitable for a diverse student population. This also offers obvious use of digital tools (such as Geogebra, Mathematica, spreadsheets etc.).

More concretely, the module consists of activities organized in three parts. In the first part we focus on the process of mathematization, mathematical modelling and recognizing the existence of functions in real life. The concept of a function is built through various non-standard activities and examples. We believe that this approach can enable teachers to support their students in building fundamental knowledge. Second part deals with conversions between different representations of functions. Mathematically, a functional dependency can be represented in many ways, using pictures (graphs), words, tables or symbols (Bloch, 2003). To make sense of the data in each representation and understand the process of conversion is one of the fundamental skills in mathematics education. The third part brings more elaborate examples of applications. In these examples we offer an approach to deepening the understanding of functions in elementary, interesting and approachable problems.

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Subject Introduction

Mathematics studies space and shapes, use of numbers and symbols and abstract reasoning. The efficiency of using the mathematical language in natural sciences, economy and other aspects of social life is remarkable. Through the process of mathematical modelling, one translates the real-life problem into a mathematical model, only to interpret the result of mathematical methods as a precise solution that makes sense in the starting context. This aspect of mathematics, as fascinating as it is, often remains hidden from students.

The topic of functions is one of the most fundamental topics in mathematics. In lower and upper secondary education in many countries it is developed gradually, with different emphasis depending on a study programme, students' interests, and culture/educational heredity. Usually, one starts with the introduction of linear functions and its connection to proportional dependency. In some educational systems, it takes years until students encounter any other type of functions or see the general definition of a function. This may cause different persisting misconceptions, e.g. the conviction that "all functions are linear". At the same time, we see curricula, in which the use of different representations of functions is not explicitly treated and the contextual problems are not realistic or motivating enough for the students.

With this module, we try to tackle these issues by considering the concept of function in a more holistic approach. We have chosen examples, tasks and activities that challenge the teachers' current way of teaching, but also offer support how to implement a more diverse viewpoint on the concept of functions. The aim of the module is to provide a more intricate web of activities through which students understand functions in a more comprehensive way. We focus on practical activities that will enable future teachers to become aware of the developmental stages of the concept of a function and to dive deeper on the important aspects that are often not presented in textbooks or not sufficiently transposed and elaborated to be appropriately situated in everyday classrooms.

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Key Competence Approach

The main focus of this module is the development of the mathematical key competence, defined by the EU as:

> *Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge. Mathematical competence involves, to different degrees, the ability and willingness to use mathematical modes of thought (logical and spatial thinking) and presentations (formulas, models, constructs, graphs and charts).*

The module also relates to other key competences such as critical thinking and scientific and digital competences. All topics in this module contribute to the mathematical competence by addressing activities that can be used to develop the competence and activities to teach the competence in classroom practices.

Interdisciplinary Approach

In this module we take on the following educational approaches:

- inquiry-based learning,
- use of real-life context,
- interdisciplinary connections between STEM subjects,
- considering girls' needs and students' diversity
- use of digital tools for deepening insight.

The topic of functions is very intimately intertwined with mathematical modelling and applications of mathematics to real world problems. In the activities we discuss problems posed in different contexts: physics, biology, economy etc. Some of the problems are situated and every-day life and the activities require both students and teachers to think critically. Through these examples the students foster the attitude that mathematics is relevant and applicable, and the teachers are provided with the potential to develop richer teaching and learning situations connecting mathematics with natural sciences and other areas. The tasks ask students to be creative and to put themselves in the role of an active citizen who takes responsibility for their judgments, creates arguments and defends their reasoning.

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Learning Outcomes

Understanding the gradual development of the concept of the function, as well as its emergence and use in contextual, scientific and every-day situations supports teachers in developing basic mathematical processes: mathematical modelling, mathematising and dealing with different mathematical representations.

Future teachers will have developed the abilities to build on students' modelling skills, to orchestrate and organize activities that support the development of the concept of functions in its multifaceted nature and improve positive attitudes towards the usefulness of mathematical models (primarily functions). Their students will be able to engage in the mathematization skills and apply them to contextual situations on the conceptual level as well as on the practical level using technology in learning and solving problems.

Learning outcomes aimed at in the module:

Topics/Learning outcomes	knowledge	skills	attitude
Observing dependencies and modelling with functions in the real world	Knowledge of the modelling cycle; concept of a function (functional dependency)	Observing variables, determining parameters, making hypotheses, interpreting conclusions	Critical appreciation and curiosity for the usability of mathematics as a tool for describing, understanding and predicting the world
Developing and using the concept of a function in different representations	Functional dependency in different representations: words, tables, formulas and graphs	Conversion between representations, choice of a suitable representation for solving a problem	Willingness and confidence to use functions and appreciation of the comprehensive nature of mathematical concepts
Recognizing and using (properties of various) elementary functions	Sequences as discrete functions; linear, quadratic and exponential functions	Recognition of similarities and differences in the properties of elementary functions	Confidence in discovering and using properties of different functions

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HE Module plan - Functions

The module consists of an introductory activity and three main parts. The total duration of the module is 525 minutes, but some activities might be opted out. The activities are aimed to foster discussion among the (pre-service) mathematics teachers preparing to teach in lower secondary school offering approaches that might be less standard in some countries. The activities might be adjusted suited for younger or older students.

The structure of the module plan is as follows:

Part 0. Introduction – Teaching functions – 25 minutes

Part 1. Building the concept of a function from context – 150 minutes

Part 2. Conversion between different representations of functions – 215 minutes

Part 3. Applying (quadratic) functions in real life – 135 minutes

Part 0: Introduction

Aim 0.1: Raising issues

Aim 0.2: Promoting mathematical competency with functions

Aim 0.3: Motivating change in teaching approach

Part 1: Building functions from context

Aim 1.1: Understanding the concept of a function and differentiate it from the concept of a relation

Aim 1.2: Understanding the parts of the modelling cycle

Aim 1.3: Recognizing functional dependencies in real-life contexts

Part 2: Different representations of functions

Aim 2.1: Building a vocabulary to talk about different aspects of functions

Aim 2.2: Making conversions between different representations

Aim 2.3: Fluency in reading and constructing graphs of functions

Part 3: Applying functions in real life

Aim 3.1: Recognizing different elementary functions and their characteristic properties

Aim 3.2: Writing algebraic expressions describing functional models

Aim 3.3: Using functions to make predictions

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Part 0. Introduction - Teaching functions

Key Competence Framework lists **mathematical literacy** as one of the eight key competences and defines it as "the ability to use graphs, charts, formulas and other presentations". The aim of these activities is to raise issues and motivate the change in teaching the notion of functions with a perspective that promotes modelling activities. Future teachers may see that this approach is oriented more towards interdisciplinary applications, but furthermore we emphasize that also the development of the concept of functions will be treated more in depth and with the focus on understanding.

Whole class	Duration: 25 minutes

Knowledge
- the notion of a linear model

Skills
- making inferences in a context based on mathematical data

Attitudes
- recognizing the value of mathematics (functions) as a tool for real-life problem solving

Activity. How do you teach functions?

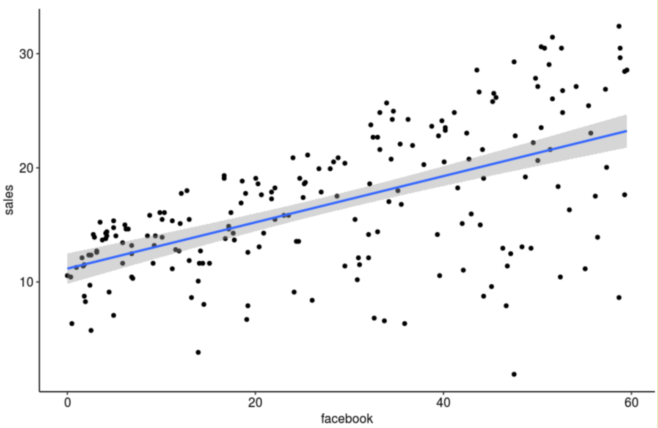
In this short activity the future teachers are invited to discuss their experience of learning about functions. The discussion can be triggered by questions:

How was the notion of functions introduced in your education? Can you describe gradual development of the concept that you have experienced? Has the approach of your teachers been oriented towards the formal and algebraic aspects of functions or towards applications of functions? In which context do we use functions, and do you have some favorite examples? Could you name some challenges in learning and teaching functions? How would you like to teach functions in the future?

The discussion could be expanded by considering the following problem:

Imagine you want to consider how your company's sales depend on the investment you make into advertising. What is the effect of Facebook advertising on the company's sales, given the effects of YouTube and newspaper advertising?

The starting point of the discussion is to realize what kind of data it is possible to obtain and how this data could be used to make any predictions. We may consider drawing a scatter plot in which **pairs of data** are showing a correspondence between sales and advertising investments on a certain platform (e.g., Facebook). The crucial conclusion is that one can observe a certain pattern, a correlation, or a trend in the data. This leads us to the notion of a **linear model** as a tool that can be used to make **predictions in the future**.



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For some students the mathematics of linear regression will be familiar, but they might only consider it as "black box" for obtaining the formula (parameters of the linear model) and may fail to explain in what way the method works or what it means that the obtained linear model is optimal. Our suggestion is to emphasize the importance of the discussion of different possible ways one could consider a linear model (e.g., by drawing a line through any two points or by drawing a line such that half of the points lie on one side of it etc.) The goal of the discussion is to raise awareness among the future teachers that it could be valuable to approach this task in an elementary way. The proposal is to engage students in creative thinking and formulation of their own arguments, before proceeding to the technically and conceptually demanding method of least squares. Even more, in programs where students are not taught the method, it might be very motivating to have this discussion about the meaning of the optimal line, which can then be calculated by software.

The task is chosen in the context that might be relevant to students and the intention is that it serves only as a motivation for the initial discussion about the concept of a model. A teacher that likes the context might notice that the data provided in the source of the task is not well-suited for the application of linear regression as the variance (scattering) of the data grows from left to right, so another set of data would be more appropriate when introducing the method of least squares to the students. Nevertheless, such examples might also provide opportunity for the discussion to what extent is the method applicable at the workplace and to what extent it is a necessity to consider (over)simplified models when learning new mathematical concepts and tools. Further development of the method is discussed in the final activity of the module as an application of quadratic functions.

Source: https://towardsdatascience.com/predicting-the-impact-of-social-media-advertising-on-sales-with-linear-regression-b31e04f15982

Part 1A. Discovering functional dependency

Mathematics is ubiquitous in everyday life, but this can remain unnoticed. It shows itself in patterns and structures, but also in certain relationships between observable phenomena. These various patterns and relationships can be described using a single mathematical concept – a function. Formally, a function is a special type of a relation, i.e., a rule that associates to each element of one set (the domain of a function) a unique element of the second set (the codomain), but this formal concept could be understood and represented in many ways, and it takes time for students to master the skill of recognizing functions and using them to solve problems.

In literature, there are a few prominent models describing the development of functional thinking, i.e., the mental process involved in understanding and using the concept of a function. The APOS theory of E. Dubinsky (1984) promotes the four stages of development: Action, Process, Object, Scheme. The four stages are similarly used by Pittalis et al. (2020) describing a function as:

(1) an input-output assignment,

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(2) a process of covariation of the dependent and independent variable,

(3) a correspondence relation,

(4) a mathematical object that can be manipulated, compared to other objects, or considered as part of a more complex mathematical structure.

Hence, before engaging in the formal definition above, students first encounter algebraic expressions (such as 2x-1) that could be thought of as machines (an action), in which a letter could be substituted by a number (the input) to obtain another number (the output). Gradually students develop the insight that the input can vary from a certain set of values (e.g., integers from 1 to 100 or real numbers), and the image of a machine expands to that of a covariance between variables.

At this stage the students describe covariational process in words as certain dependencies between two quantities (e.g., proportionality) and gradually develop other ways of representations such as tables, formulas, and graphs. They might still not be aware of the importance of the choice of the domain and the codomain and think of functions only in terms of their association rule (e.g., $y = 5x + 2$ or $f(x) = 2^x$). This level of understanding the concept of functions is not yet formal, but it is certainly rich enough to allow for using functions for problem solving (modelling) and for significant student engagement. On the other hand, in some countries this stage is somewhat rushed and there is a missed potential to develop mathematical concepts intertwined with the development of competencies to apply these concepts. The whole module focuses on filling this gap by emphasizing the elementary use of functions in context. For this reason, almost all the activities are based on the situations (or problems) that are suited to be considered in lower secondary education but could also be adapted or (re-)used at a higher stage. At the higher secondary level, one would continue by developing a more formal definition function (e.g., emphasizing the role of the domain in the context of bijectivity of a function and finding inverses) and even to more elaborated uses of functions in (pre-)calculus, where the function is considered as an object that could be manipulated (e.g. integrated, multiplied with another function or whose graph could be transformed by using a geometrical mapping). This part of the module consists of three activities that bridge the transition between the first three stages, while the other parts of the module provide activities that provoke deeper questioning of the concept and examples of applications/modelling.

We end this short overview by pointing out some limitations. For example, D. Tall (1999) reflects on APOS theory and questions the idea of primacy of action before object invoking neurological findings about the cognitive development (such as the existence of specialized parts of brain for visual perception). Motivated by such considerations, we emphasize that the development of functional thinking will not be the same for all students and does not necessarily follow the above order of stages. Also, some students will develop various connections more quickly and different students will find different representations more intelligible. In the digital era, we may certainly encounter new student learning paths that might be oriented more towards visualizations (e.g., use of graphs over reading text) and use of technology both as a tool in solving problems or as a source of new knowledge.

Individual or pairs	Duration: 3 x 20 minutes

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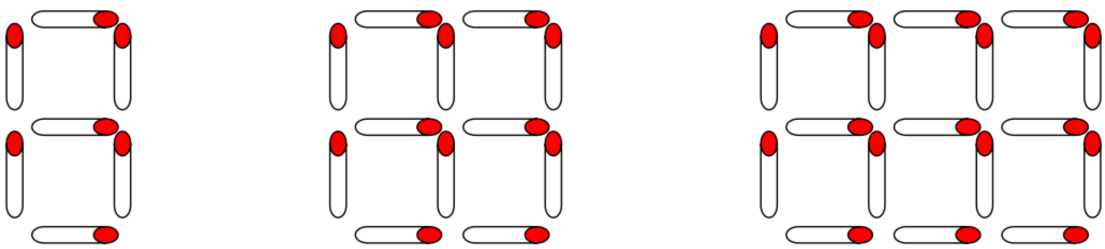
Knowledge
- writing a functional rule describing a sequence
- definition(s) of a function
- differentiating between the notion of a function and a relation
- formulating different stages of developing functional thinking

Skills
- recognizing a functional dependency in real-life situations
- inductive reasoning

Attitudes
- acknowledging the fact that functions are ubiquitous in life
- appreciating the gradual development of mathematical knowledge

Activity. Finding a pattern

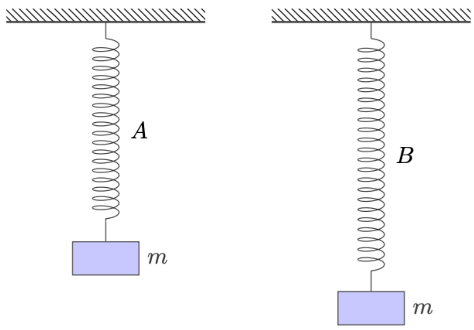
In this activity the participants are given a (short, finite) sequence of shapes consisting of some number of elements (e.g. matches). The shapes grow in size and form a pattern that is to be a discovered. The task is to find the number of matches in the n^{th} shape (for a concrete given or general number *n*) and the answer (5n+2) can be reached in many different ways. With future teachers, the discussion is led how to form different sequences of shapes and with which aims. What is the role of inductive reasoning and at what age is the task appropriate? What are the different **strategies** and **patterns** that one can use to solve this task? Further discussion can be led about the different didactical potential that the task when presented to primary school students (introducing **algebraic expressions**) or to secondary school students (the difference between a **discrete and continuous model/function).** Such discussions can also foster raising awareness of the transition from input-output perspective to covariational way of thinking.



Activity. Hanging springs

This short activity is meant to trigger the discussion about the use of symbols in mathematics education and the extent to which students rely on procedural approaches over contextual and conceptual understanding. It revolves around the following task.

Two hanging springs A and B have lengths that can be calculated using the formulas $y = 20 + 0.4x$ and $y = 14 + 0.6x$, respectively, where y is the length of the spring (in centimeters) and x is the weight (in grams) hung from the spring. Use the given equations to explain how it is possible that the two springs are equally prolongated when equal weights are hung on both springs. Draw a diagram to show this.



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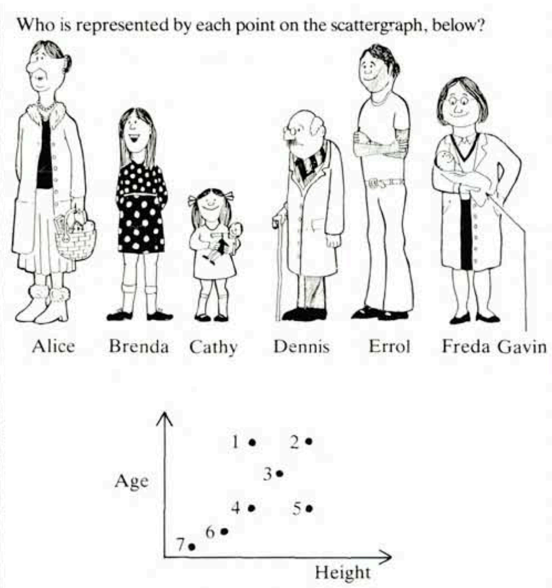
To solve the task, a student should understand the meaning of the **equation** as a covariational dependency. The equations can be considered separately as rules showing how the length changes as one changes the weight hung on a specific spring. The coefficients in the equation are interpreted as the initial length of the spring (no weight) and the spring constant. The solution is based on the interpretation that both the value of x and the value y are the same, which is possible only for one specific pair of values obtained as the solution of the system of two linear equations. The task provides the potential to discuss the use of language and typical students' misconceptions (e.g. difference between an equation and a function). As the task is set in a physical context it can be considered from the perspective of modelling, but the mathematical model is already given, so it remains to focus on **interpretation and making sense of the given model**.

Activity. Relations vs. functions

Functions are special relations and functional dependency is only a special type of dependency. The participants are invited to make up different examples of **discrete and continuous relations**, some of which will be functional and some not. E.g. the relations about mothers and children, distance of a person walking in some direction from a given point, numbers and divisibility, time-dependent motion etc.

Related to the given task, the students are asked to discuss the different strategies by which the students could reason, the possible misconceptions stemming from the fact that the height denotes the horizontal axis, and different ways of describing functions. The discussion concludes with the clarification of what is the formal definition of a function (as a special relation) and what are the less formal approaches appropriate to students. Note that the activity may be also connected to the discussion about the linear regression presented in the first and the last activity of the module.

Source: The Language of Functions and Graphs, Shell Centre for Mathematical Education Publications, 1985.



Activity. Discovering functions in real-life

In this module we will explore the notion of a function which is one of the main tools in mathematics to describe a covariational situation. This brief activity is an exercise in **divergent thinking and creativity**. The participants need to come up with many functions that describe a phenomena or relationship that they can see around them in that moment. The discussion should direct this creative stream of examples towards the consideration of the aims of lesson and the objectives a teacher wishes to achieve with the students. So, the most important question is: Why do you think that this is a valuable example and to which outcomes does it contribute?

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Part 1B. Functions as mathematical models in different contexts

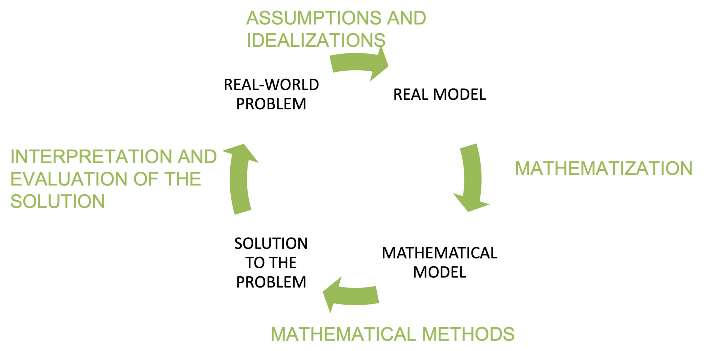
Mathematization, as it is used in OECD/PISA, refers to the organization of perceived reality through the use of mathematical ideas and concepts. It is the organizing activity according to which acquired knowledge and skills are used to discover unknown regularities, relationships and structures (Treffers and Goffree, 1985). This process is sometimes called **horizontal mathematization**. It requires activities such as:

- identifying the specific mathematics in a general context; – schematizing
- formulating and visualizing a problem;
- discovering relationships and regularities; and
- recognizing similarities between different problems.

As soon as the problem has been transformed into a mathematical problem, it can be resolved with mathematical tools. That is, mathematical tools can be applied to manipulate and refine the mathematically modelled real-world problem. This process is referred to as **vertical mathematization** and can be recognized in the following activities:

- representing a relationship by means of a formula;
- proving regularities;
- refining and adjusting models;
- combining and integrating models; and
- generalizing.

Solving problems through mathematical modelling is theorized in the literature in various ways. While some approaches consider modelling an open-ended activity, other consider finding (some, at least partial) solution to the problem an end of the process. Furthermore, there are iterative approaches that consider improving the process in a spiral or cyclic organization. We consider an approach of the **modelling cycle** consisting of four stages (Kaiser 1995, Blum 1996). Starting with a real-world problem situation we first need to make certain simplifications and assumptions. In that way we reach a *real model* of the situation. Next, through the process of mathematization we build a **mathematical model** based on equations, variables and functions. The third stage consists of solving the problem using mathematical tools. Finally, the solution is interpreted in the starting context and we evaluate the correctness and usability of the obtained solution.



Pairs and whole class discussion	Duration: 2 x 45 minutes

Knowledge
- recognizing different stages/phases of the modelling cycle
- differentiating between a real-world situation, its real model and its mathematical model
- differentiating between horizontal and vertical mathematization

Skills
- discussing and recognizing different aspects/parameters of a real-world situation

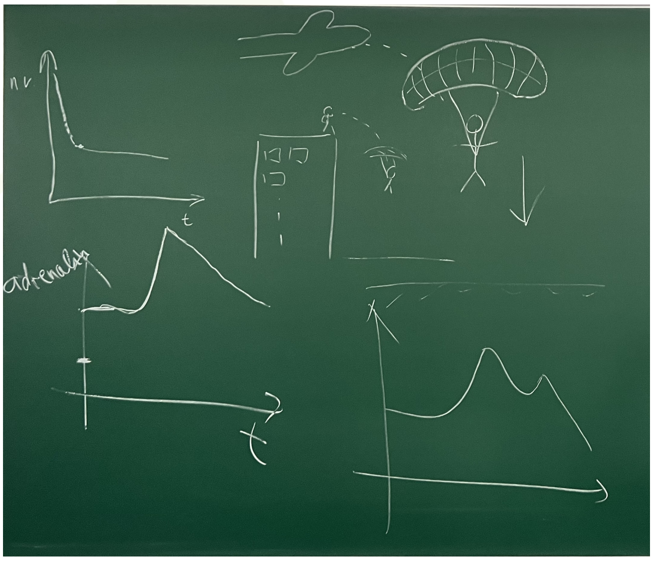
Attitudes
- accepting functions as a tool for real-life problem solving

Activity. Parachute jump

Consider a situation in which a person jumps with a parachute from a flying airplane. This situation that could be used to organize rich discussions in a three-fold way. In this part of the module, we use the situation to discuss the first part of the modelling cycle or the horizontal mathematization. The future teachers are introduced to the idea of a real model and asked about the possible **goals, assumptions and parameters** that are important to be considered when trying to model the jump. At a different time, the situation might be used to practice more elaborate mathematization such as describing the situation with functions and graphs (qualitatively) and finally to develop the theory of linear ordinary differential equations and to apply to obtain the jumper's trajectory, but for now we only consider the value of the situation to discuss physical aspects.

The activity emphasizes the need to discuss the problem in phases, starting from every-day experiences and verbal descriptions, towards physical assumptions and eventually the mathematical model. We discuss with the participants what kind of relevant data the students might consider. Would they think about different motivations for the jump? Which quantity will they model (height, speed or something else)? Will they consider the situation **critically,** e.g. by considering safety measures and looking for real data? How to shift the focus to the physical and mathematical aspects? Will the students realize how crucial it is to consider the size of the parachute and the drag force as the main components of the model along with the gravity force?

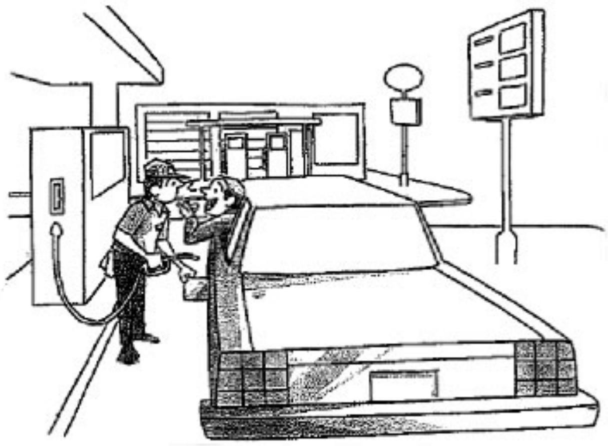
As a first step towards the description of the situation students may draw simple pictures. This process of simplification is an important part of developing modelling skills and could be followed by a discussion about the differences between jumping from an airplane (hence starting the jump with some horizontal speed) and jumping from a building (almost like a free fall from a lower height).



Drawing pictures also provides opportunity to discuss the difference between a sketch of the jumper's trajectory and a graph describing some functional dependency in an abstract coordinate system. Once considering the situation with the students which are more fluent in using graphs of functions, the discussion can be directed in a more **creative** way by asking the participants to consider various graphical representations of the phenomena related to this situation. For example, asking questions about the dependency of the adrenaline level in the blood of a jumper during the jump. If there is plenty of time, future teachers could also investigate **mathematical descriptions** (speed-time graph, differential equation etc.) of the fall on the internet and try to make sense out of it.

Activity. Which gas station?

Another modelling situation to consider concerns buying gas at two different gas stations. One gas station has higher prices, but it is on our typical way from home to work. The second is not on that way and requires a detour but has cheaper gas price. Which parameters would you consider? Can you write a function that calculates the efficiency of buying gas with a detour? Can you make the situation more concrete and draw a graphical representation?



"Just enough to get me across the street to the cheaper station."

(Parade, 12 Nov 2005)

The activity could continue by discussing a concrete version of the task and focusing more on the mathematical aspects of the model (setting up various equations and solving them, reformulating a problem as an optimization problem of a certain function). Which **phases of the modelling cycle** can be omitted (because they are already given) if we deal with the following task?

Frank usually buys gas at the station on his route between home and work at the price of 2.00 EUR per liter. At a station located 5 kilometers from his route gas is sold for 1.80 EUR per liter. Does it pay off to make a detour from his usual route to buy cheaper gas?

This group of activities provides an important scheme for mathematical modelling, which is exactly the use of mathematics in solving interdisciplinary problems. Each stage involves critical thinking. In the first stages we need to consider the quality of our data and probably consider multiple sources to understand the problem better. In the later stages, we need to evaluate the solution and reflect on the quality of the problem-solving process. The examples show relevance of mathematics in physics and economy, so they are true cases of **interdisciplinarity**.

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Part 2A. Different ways to represent functions

A functional dependency can be described in words, using symbols, tables (pairs) or graphs. We say that we are using different *representations* (Duval, 1995). In the first part of the module we discussed the process of identification and recognition of functions in real-life contexts. We have already used some of the representations, but the focus was on building a concept of the function regardless of a representation, which means that students are expected to have their own conceptions of the functions, or to understand the concept only in a concrete situational context based on the seen examples. In this part we make each of the representations more explicit and through a sequence of activities we build on the skill of making conversions between these representations. We start from every-day examples in the verbal representation (words) which are described then in other representations and slowly progress towards other conversions (e.g. between graphs and formulas).

Individual or pairs	Duration: 4 x 20 minutes

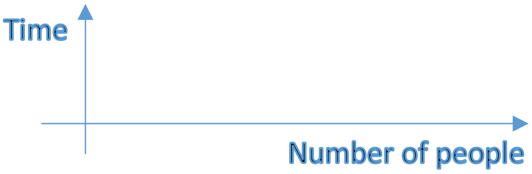
Knowledge
- representing functions with symbols
- drawing graphs based on tables, function rules and verbal descriptions
- reading and interpreting graphs of functions in words
- solving equations and interpreting the solution in the context

Skills
- making sense of conversions between different representations of functions
- choosing appropriate representations based on the type of a problem
- evaluating the efficacy of approaches based on different representations

Attitudes
- appreciating the comprehensive nature of functions (and mathematics in general)
- building a conviction that using mathematics makes sense

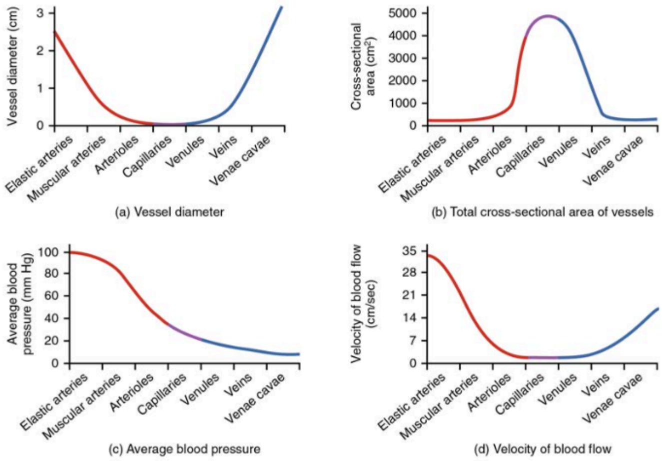
Activity. Sketching graphs based on verbal descriptions

Comprehensive reading and expressing mathematical relations in words are very underestimated skills, that should gain more attention in schools. Participants are asked to share how would they encourage students to engage in the more precise verbal expressions of mathematics. This activity relies on the experience of individuals with a typical task that involves **(inverse) proportional thinking**. The task itself asks the students to compare their solution with peers and to be critical about it. It is expected that the students will anchor themself in the conception that a graph of a function should be a line, i.e. to insist on 'linear thinking'. By overcoming this obstacle, the students should realize that there are many useful functions which are not linear. The students are asked to use the following axes to illustrate the sentence "*The more people we get to help, the sooner we'll finish picking the strawberries*." The graphs should be compared and discussed among peers.



Activity. Interpreting graphs

Graphs of functions can be very informative, but one needs to know how to read them. The task (taken from an anatomy course) is to read information from a graph in which the values on the x-axis are given by systematic vessels in a human body.



Graphs should be **explained in words** and we note three levels of complexity. On the first level, the student reads a single graph, e.g. "the capillaries have the smallest diameter". On the second level, the student draws conclusions by using two graphs, e.g. "the capillaries have the smallest diameter, but there are so many of them that they make the most of the cross-sectional area". On the third level, the student discusses the graphs from a more formal mathematical perspective, e.g. the graphs are continuous, although there are seven types of vessels written as the labels on horizontal axis. This leads to the interpretation that the order in which the types of vessels are written presents a continuous spectrum, a fact that can be more easily explained by an expert in anatomy. Hence the example may again trigger the discussion about discrete vs. continuous models, and it can show the importance of interdisciplinarity.

Source of the picture: *Blood Flow, Blood Pressure and Resistance*, Anatomy and Physiology II, course on Lumen Learning. Link: https://courses.lumenlearning.com/suny-ap2/chapter/blood-flow-blood-pressure-and-resistance-no-content/

Activity. Comparing growth

In this activity the future teachers may become aware that the **arithmetic and geometric progression** are discrete analogs of linear and exponential function. In general, many people will be aware of the idea that "the exponential function grows faster than the linear", but this activity challenges this statement and asks the participants to delimit more precisely when it holds. We are given the following task:

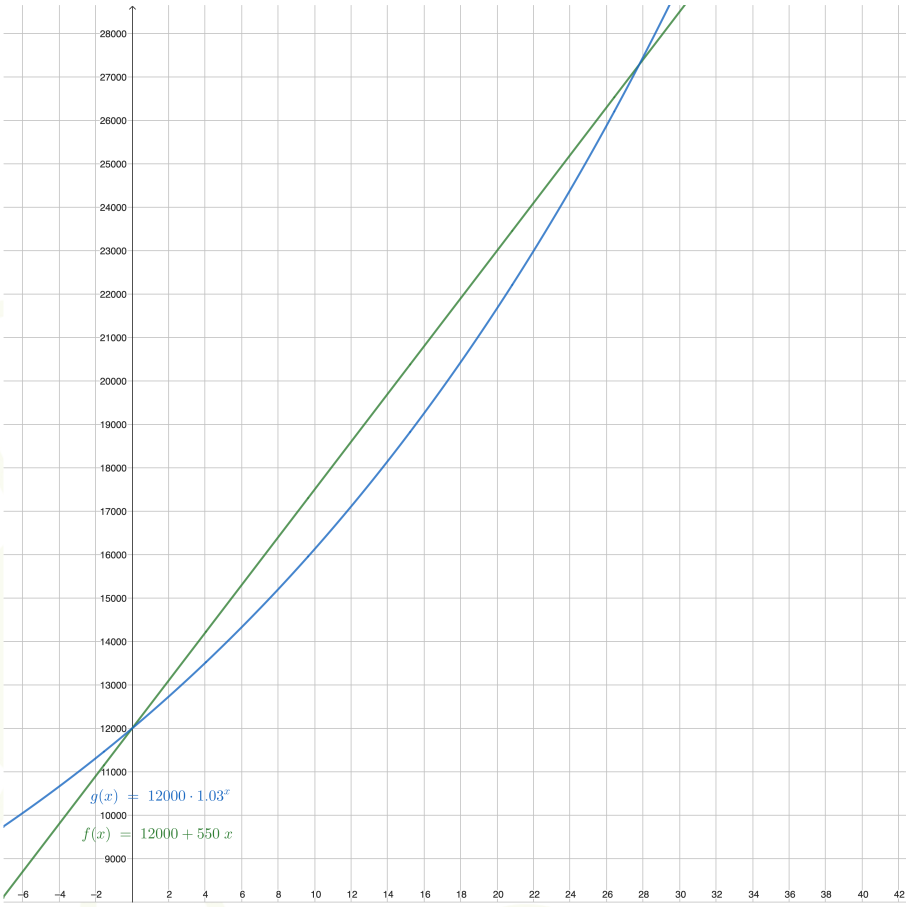
In 2018 Andy had 12000 EUR. Each year he added 550 EUR to the account. In 2018 Barbara had 12000 EUR. Each year she received interest rate of 3% on her investment (as she kept it in the account). Calculate the amount of money they each year until 2030, write a general formula for the amount they will have after x years and compare the growth.

Already on the level of discovering patterns, students may fill in the following table and perhaps find more values or the general rule describing the dependency of the amount of money on time.

x	Andy (12000+550x)	Barbara ($12000 \cdot 1.03^x$)
5	14 750	13 911
10	17 500	16 127
15	20 250	18 696
20	23 000	21 673
25	25 750	25 125
30	28 500	29 127

The data can be presented as a scatter graph, but also a general rule may be written algebraically, and the graph of a real function can be drawn. From these graphs, one can conclude that **the exponential growth will be slower in the short-term, but much faster after a certain moment in time** (in this case around 27 years).

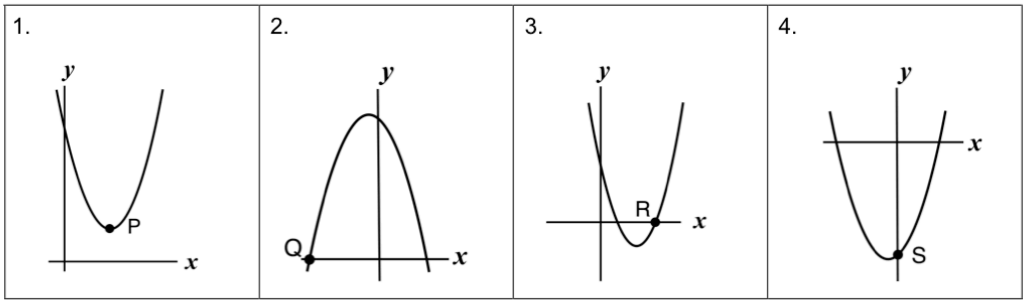
Finally, the context supports the development of the exponential functions beyond integer exponents. For example, it is natural to ask a question: What interest rate on a monthly basis will provide the yearly interest rate of 3%? A simpler question is to consider a semi-annual basis. If the required interest rate is *r* %, then we have $(1 + r)^2 = 1.03$, from where we find the value of *r* directly.



Activity. Connect the graph and the formula

A function graph can be drawn by determining a few points and then connecting these points, but it can be also drawn by **transforming the fundamental graph**. To prepare for drawing the graphs in different ways, one should see different algebraic ways of writing functions. In the task, the equations should be connected to graphs and the more ambitious explanation concerning transformations of graphs can be used, e.g., arguing about the graphs based on the zeros (e.g., B and D are 4 and 2) or the extremal points (C is 1).

A. $y = x^2 - 6x + 8$	B. $y = (x-6)(x+8)$	C. $y = (x-6)^2 + 8$	D. $y = -(x+8)(x-6)$



The activities are related to other key competencies such as literacy. Three of them are given in the contexts that are familiar to the students, so they may also activate their own experience in reasoning about the solutions. The final task is given completely in the mathematical context, but one can insist on students' argumentation to foster critical thinking and justification of the choices made.

Part 2B. Graphs from a higher standpoint

Graphs are very informative as they provide visual representation of the whole function at once (object perspective) and can comprise many different aspects of a real-world situation. As these aspects might be similar and close, there are many possible misconceptions in their interpretation. In this part we consider one misconception which arises in tasks where the context is described geometrically (e.g., a shape of a track) and it is required to describe it with a graph of a motion based on time. The main misconception to be dealt with is that the geometrical properties of the context should be reflected in the geometrical properties of the graph. Introduction to mechanics (physics) usually deals with the situations in which the motion can be described by the time-dependency of the position, speed, or acceleration. The graphs representing these dependencies are quite different and to answer questions about one aspect based on another is very challenging. Mathematically, more complex situations could be of course described in terms of derivatives and integrals, but the selected examples that we present could be explained without these terms and are hence very suitable for students at the pre-calculus level. These examples show that it is possible to discuss and to build student's intuition about the fundamental ideas behind concepts from calculus (such as rate of change or accumulation) at a younger age.

Think-pair-share (think individually, discuss in pairs, share with everyone)	Duration: 3 x 45 minutes

Knowledge
- drawing graphs based on geometrical description of a real-world situation
- making distinction between geometrical description of a real situation and it abstract graphical representation
- reading the speed of change from the graph

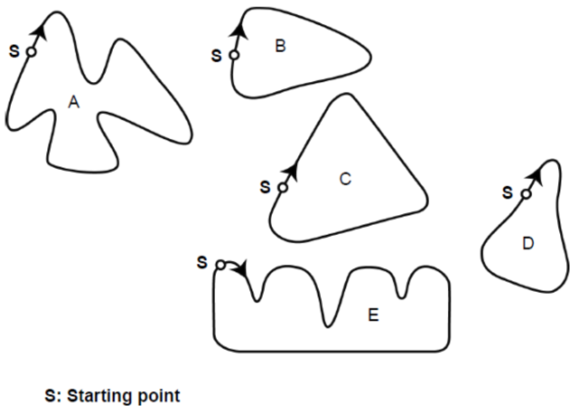
Skills
- solving problems based on drawing and interpreting a graph of a function
- dividing a complex problem into smaller parts
- describing the solution to a problem in own words

Attitudes
- relating one's own experience to the solution of a mathematical task
- making decisions based on information and mathematical argumentation

Activity. Racing car

The activity is based on a famous task from PISA-assessment, which has led to various research papers and has appeared in different teaching resources. In the task the conversion is made **from the geometrical description of a track to a graphical description of the speed of a racing car** that drives along this track (for many different tracks), or vice versa to find the track which corresponds to a given speed-distance graph.

The task is challenging because both descriptions are given by curves, but their geometrical properties (curvature) are subtly connected. The solver of the task is required to think about the driver's perspective and the basic rules of driving a racing car. A typical student's mistake is to say that the graph will be curved when the track is curved, and vice-versa. A more appropriate reasoning is to say that the speed remains constant if the curvature of the track does not change (e.g. around the circle the car drives at the maximal speed that the curvature allows), and that the car speeds up when the curvature is decreasing and that the car slows down when the curvature is increasing. Teachers or student teachers should think about students' difficulties and design tracks that would challenge students' reasoning.

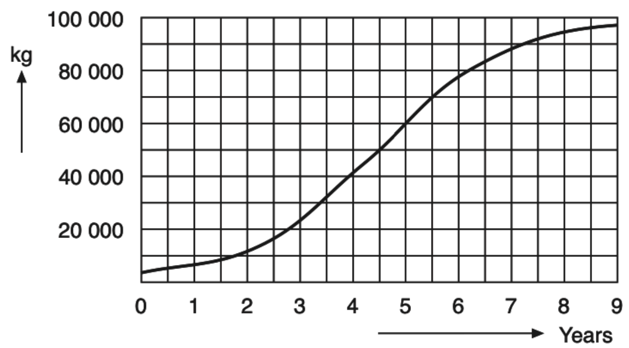


Source of the picture: Strohmaier et al. *Different complex word problems require different combinations of cognitive skills,* Educational Studies in Mathematics 109(3), Springer, 2022. English version of the item Racing Car (M159Q04). Adapted from PISA Released Items – Mathematics (p. 33), by OECD (2006). Copyright 2006 by the OECD. Used under CC BY-NC-SA 3.0 IGO

Activity. Fish growth population

The following task is another example showing the highest class of competency (mathematical thinking, generalization and insight) tested on the PISA exams of mathematical literacy. Students need to mathematise the situation and draw conclusions using a graph of a function beyond the typical skill of reading of points of graph. We use this example to show that functions are the best way to describe the growth of a population over a period of time. To solve the task, one needs to realize that it is optimal to catch fish in the time period of one year in which the **graph is the steepest** because in that way the yield will be maximal (or more formally, one can say that this is the interval in which the function has the biggest derivation). It happens between years 4 and 5.

Some fish were introduced to a waterway. The graph shows a model of the growth in the combined weight of fish in the waterway.



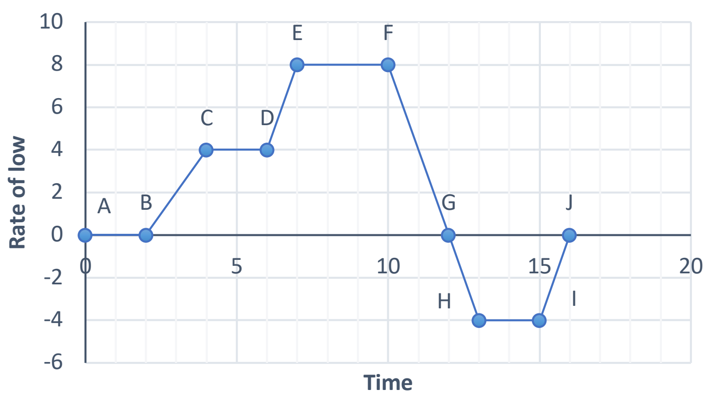
Suppose a fisherman plans to wait a number of years and then start catching fish from the waterway. How many years should the fisherman wait if he or she wishes to maximise the number of fish he or she can catch annually from that year on? Provide an argument to support your answer.

Source of the picture: Measuring Student Knowledge and Skills - A New Framework for Assessment, OECD Programme for International Student Asseessment, 1999. Link: https://www.oecd.org/education/school/programmeforinternationalstudentassessmentpisa/33693997.pdf

Activity. The rate of flow

We discuss an important situation in which the students need to reason abstractly. The figure shows the rate at which the water flows into or out of a container at different times.

To discuss when the water volume in the container increases or decreases (the fastest), one needs to make the connection between the flow rate in time and the total volume. This can be simply commented in terms of the area (or more generally the integral of a function).



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The total volume of the water at a certain moment is given by **the area between the graph and the x-axis** until that moment. The teachers need to address the fact that if the graph is below the x-axis, the area is subtracted (i.e. should be considered as negative). This is explained by the fact that negative speed means that the water is flowing out of the container. The task is inspired by a Swedish high-school textbook.

This collection of activities presents more complex notions and uses of functions. It requires the activation of critical thinking, to make sense of the situation and the result, and to understand the value of the functional approach. All the tasks in this collection are given in a context outside of mathematics. They show the application of mathematics and the need to reason based on the understanding of the context when solving such tasks. We find that it is also important for the teachers to see that complex mathematical reasoning is possible without (i.e., before) formally introducing many mathematical concept (from calculus).

Part 3. Quadratic functions in real-life

The quadratic functions are quite often introduced with formulas and their applications are shown afterwards, but there are many natural contexts in which they appear. In this part we emphasize that a quadratic function is characterized by its property that the differences are linear. This property is discovered by students in calculus in terms of derivatives, but with our example we show that this can be discussed earlier and used even before the students know quadratic functions to break the misconception "all functions are linear".

In the second task we deal with a very important, but challenging skill to mathematically model real-life situation described in words. The model is an algebraic expression, i.e. a function which needs to be optimized. For students, the skills of writing formulas down, or using and interpreting formulas may be very hard and is quite often done without comprehension why the method actually works.

Individual work and classroom discussion	Duration: 3 x 45 minutes

Knowledge
- characterizing quadratic functions as functions whose first order differences are linear and second order differences are constant
- optimizing (finding the minimum or the maximum) of the quadratic function
- using the result of optimization to solve real-world problems
- deriving the formulas based on the least square method

Skills
- organizing data and fluency in calculations
- using technology to make calculations and predictions
- solving and interpreting solutions to optimization problems based on algebraic approach

Attitudes
- using mathematical calculations to justify decisions related to civil education
- acknowledging the predictability power of the scientific approach to problem solving

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Activity. The braking distance

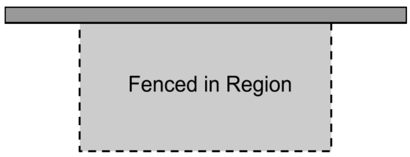
Source: Pixabay

Speed limits are a nice topic to **critically discuss responsibility** among young drivers (and drivers to be). So, we propose a task in which the students need to investigate the dependency of the braking distance on the speed in the moment when the car starts to brake. This dependency is quadratic, which might not be known to all students. Some students may derive a formula based on other more familiar formulas from physics, but we set up the task in the way that this is not necessary, and all conclusions can be done mathematically. The task is given in terms of **second differences** which are constant, and the students need to discover, probably to their surprise, that the desired function is quadratic. From the quadratic dependency it follows that a slightly bigger speed will imply substantially longer braking distance.

Source: Braking distance scenario, Erasmus+ Project MERIA, 2019.

Activity. Festival area

Quadratic expressions occur naturally when we talk about areas. To study the quadratic function, we can set up a situation in which the area of a certain fenced region varies as we change the dimensions of the fence. To make the problem simpler we can assume that the shape of the site is rectangular. Further variations are possible to make the problem more interesting and to emphasize the algebraic approach over argumentation based on symmetry. One such possibility is to let one side of the area to be a wall instead of fence. Context can be chosen also in many ways, e.g. that the region represents a festival site or a yard for sheep. Finally, we may expect **different student strategies**, from guessing, to drawing many data points which are analyzed by quadratic regression (with the assistance of technology) or to setting up an algebraic expression and optimizing the function using algebra or calculus. All these parameters are to be discussed with (future) teachers with the aim of recognizing how rich this set up really is.



In the task we fix the total fence that can be used for three sides of the rectangular festival area. The activity emphasizes the variability of one side length and the nature of function as a rule that expresses how one observable changes according to the other. It makes foundations to **use algebra and differential calculus in the optimization problems**. Formulating a functional rule $A(x) = x(P - 2x)$, where P is the perimeter of the fence, while $A(x)$ is the area corresponding to the width x, shows to be a very challenging issue for students, which is often underestimated by the teachers. The participants are invited to reflect on the students' difficulties in setting up the algebraic expression. Could we have approach this with a regression strategy? Also, the question of interpretation of the model includes the discussion about the domain of function A, i.e., about the allowed values of x.

Source: Festival site scenario, Erasmus+ Project TIME, 2022.

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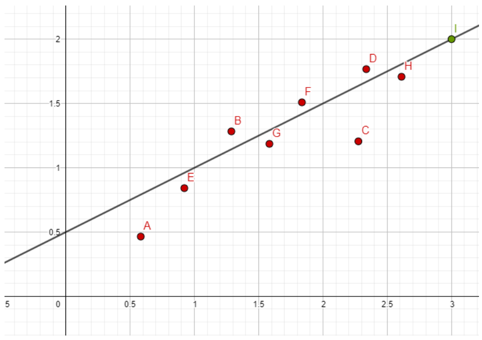
Activity. Linear regression



Linear regression is a well-known method for finding an optimal linear model that describes a given discrete data set representing a dependency or correlation between two observable quantities. We often see that students are given complicated formulas, but not enough attention is dedicated to comprehensive understanding of the method. First of all, students need to realize how the dataset is represented and what it means for a function to be a linear model of the dataset. Next, we discuss the meaning of '**optimal**'. A line is represented by its equation $y = ax + b$, and all the information is actually given by the coefficients a and b. We need to define the criterium based on which we will say that a pair of coefficients (a, b) will be optimal. Many criteria could be considered and one could provide many reasons why we take the pair for which the expression

$$F(a,b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

is minimal, where (x_i, y_i) are the points from the given dataset. The reasons are subtle and are related to statistical way of thinking (the above expression represents the variance of the given data around the mean) and to technical advantages (such a function is derivable in each point). Finding the optimal solution can be done in an elementary way. Based on the context, or by taking for granted, we assume that the point whose coordinates are mean values of the data in the dataset lies on the graph, or equivalently that the sum of the 'residuals' is zero. This enables us to reduce the number of variables by one and hence obtain a single-variable function. This function is quadratic in a, and hence the optimal parameter is obtained as an application of the quadratic function.

In this part we suggest using two examples that have a lot of didactic potential, in particular for students' autonomous investigation. In this way we foster **inquiry-based approach** to mathematics education and ask students **to think critically** about situations in context. Of course, the braking distance is related to physics, but it also carries an important aspect of civil responsibility. The festival site is a context familiar to students and this supports students in raising further questions and observing relevance of mathematics. In the third example we return to the problem of predicting using linear regression. Regression line is obtained using the method of least squares based on the application of the quadratic function and it has many uses in various fields.

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Materials and resources

Presentation. Is it Mathematics? Modelling with functions.

The presentation is based on the activities Parachute jump, Fish growth population and Racing car. It supports orchestration of a workshop in which students (future teachers) learn about the modelling cycle and graphical descriptions of different growths.

Worksheets. Functions.

The worksheets provide support for various activities described in the module: Finding a pattern, Relations vs. functions, Parachute jump, Which gas station?, Drawing graphs based on verbal descriptions, Interpreting graphs, Comparing growth, Racing car, Fish growth population, The rate of flow and Linear regression.

Readings.

For further reading we suggest the list of references given below.

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Evaluation

Understanding the modelling cycle

Evaluation goal: Knowledge on the modelling cycle, terminology related to it and the teaching competence related to recognizing suitable tasks.

What is mathematical modelling? Can you give an example?
What are the stages of the modelling cycle? What happens in each stage?
Consider a typical textbook task given in an every-day context. Which parts of the modelling cycle are already given, and which are left to the solver of the task?

Comparing experience

Evaluation goal: To raise awareness about the different teaching/learning pedagogies and the role of inquiry-oriented tasks in the given activities.

In this activity participants (teachers or future teachers) compare their own teaching practice (or learning practice while they were students) with the activities described in the module. Participants should discuss in groups of 3-4 and try to make a list of similarities and differences among the approaches influenced by the choice of tasks and activities.

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Critical reflection on functions

Evaluation goal: to explore ways to assess students critical thinking while solving a complex task in a real-world context. See IO1 for information on the rubric for Critical Thinking and its underpinning.

This activity is based on the activity called Racing car. The students have the task to study different racing tracks and associate them with the graph that describes the dependency of the speed along the track. How would you evaluate students' approach and reasoning to the task? Use the following rubric.

Neutral level

- *The naïve solution that the graph is curved at the places where the track is curved*
- *Misinterpreted variables on the axes of the graph*
- *No perspective of the driver of the car*
- *Sloppy graphs*
- *Drawing obviously wrong graphs*

Basic level

- *Applying a strategy that works only for some tracks that are similar to a shown example*
- *Explicitly linking the curvature of the graph with the curvature of the track*
- *Neat graphs, although not always correct*
- *Description of the reasoning in written text*
- *Evident change of a graph during the solving process*
- *Comparing the graph and concluding that it is or is not correct*

Proficient level

- *Evidence of at least two ways of explaining the shape of the graph,*
- *General patterns such as constant curvature of the track leads to constant speed*
- *Providing explanations considering practical and psychological aspects of a race*
- *Neat and correct graphs*
- *Confidence in the graphs and the reasoning behind them*

Expert level

- *Making up and discussing different types of tracks and graphs, not given by any source*
- *Providing explanations for efficiency of the method that directly connects the curvature of the track and the graph*
- *Expressing the value of functional thinking*
- *Discussing the difference of time-dependence and space-dependence of the speed*
- *Discussing possible misconceptions or pitfalls that beginners might encounter*

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